THE UNIVERSITY OF ALBERTA

TSYPKIN AND JURY AND LEE STABILITY BY ROOT LOCUS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled Tsypkin and Jury and Lee Stability by Root Locus submitted by Sydney John Deitch in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The Tsypkin and Jury and Lee stability criteria are useful for obtaining constraints for certain nonlinear sampled-data systems.

The Tsypkin criterion assumes no knowledge of the nonlinearity, while the different Jury and Lee criteria assume varying degrees of information about the nonlinear element. In this way the Tsypkin criterion produces a more conservative result than the Jury and Lee criteria.

These criteria are extremely laborious to evaluate without the aid of a digital computer.

In this thesis a method of evaluation using root locus techniques is developed. Without a computer the different criteria can be evaluated quickly and with reasonable accuracy. The method is based upon the root locus interpretation of the Popov criterion for continuous-data nonlinear systems, developed by H. D. Ramapriyan, M. D. Srinath, and M. A. L. Thathachar[1].

ACKNOWLEDGEMENTS

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CHAPTER 1 INTRODUCTION

The nonlinear sampled-data system under consideration is shown in Fig. 1.1.

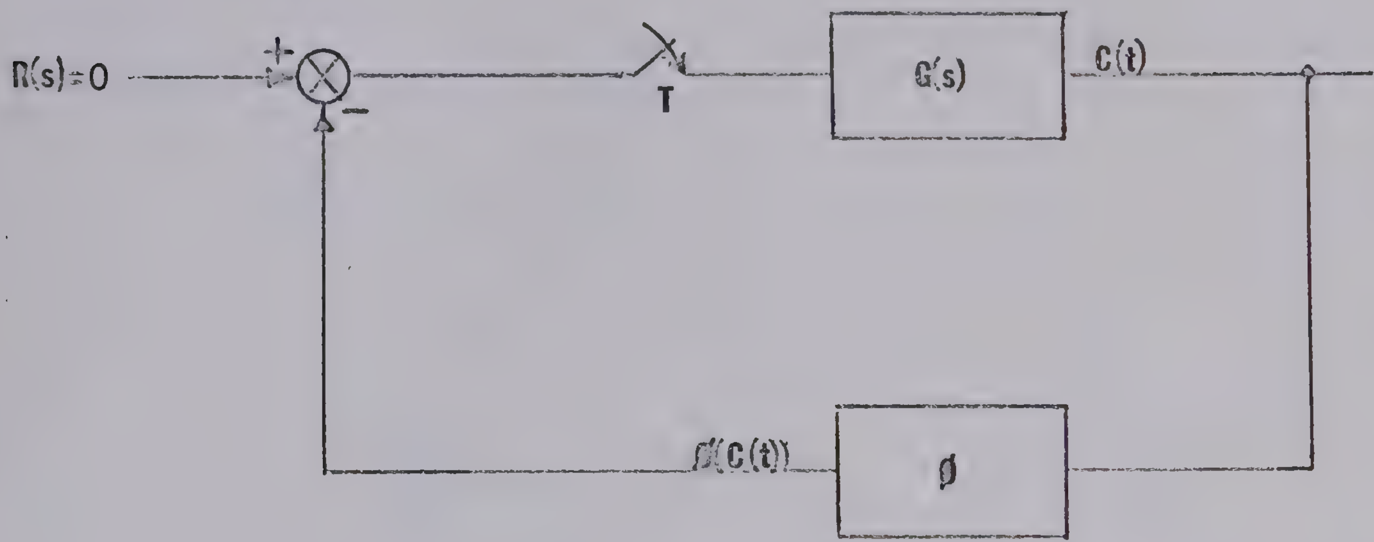

FIGURE 1.1. TYPICAL SYSTEM CONSIDERED

In the above figure G(s) represents the linear portion of the system, and the symbol ø represents the nonlinear element.

The four stability criteria which will be investigated are all valid for systems of the type shown in Fig. 1.1. A compensation method based on the Tsypkin criterion utilizes a filter which cascades with G(s).

The stability theorems discussed are expressed in the z-domain. In order to investigate whether the various criteria are satisfied for given systems it is necessary to evaluate them for $z = e^{sT}$ on the unit circle in the z-plane, or in other words for $|z| = 1$. To obtain the root locus method of evaluation it is necessary to use a bilinear transformation ($z = r+1/r-1$) which transforms the unit circle in the

z-domain into the imaginary axis in the r-domain. After this the criteria are manipulated in a form which is transformed again by use of another transformation $x = -r^2$.

CHAPTER 2 THE TSYPKIN CRITERION

2.1 Evaluation of the Root Locus Method

For systems of the type shown in Fig. 1.1, and nonlinear elements with the properties and form as shown in Fig. 2.1, stability is guaranteed if[2]

$$\mathrm{Re}\left[G(z)\right] + 1/K > 0 \quad \text{for } |z| = 1. \tag{2.1}$$

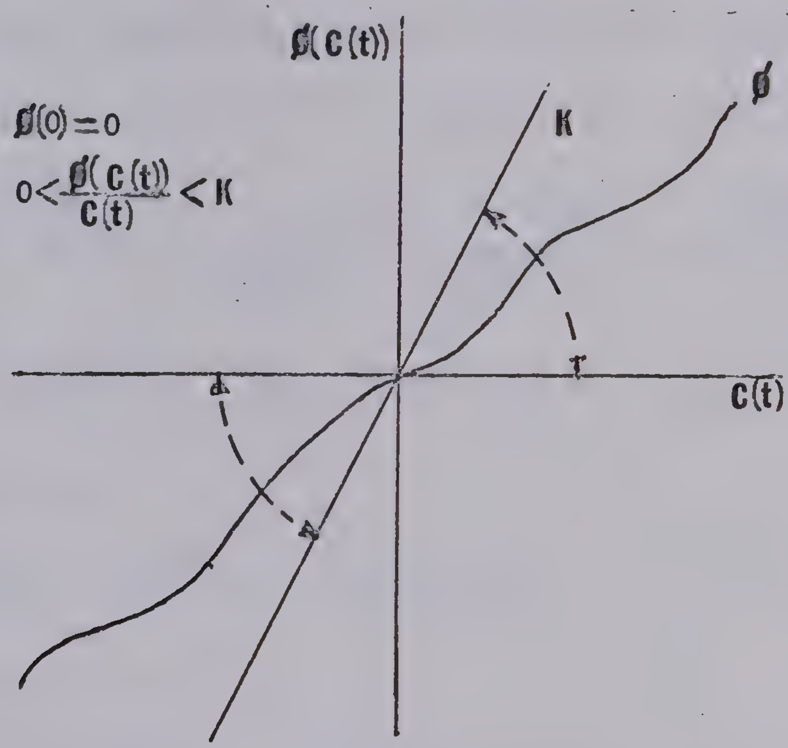

FIGURE 2.1 TSYPKIN CRITERION

By use of the bilinear transform (2.1) becomes

$$\mathrm{Re}\left[G(r)\right] + 1/K > 0 \quad \text{for } -\infty \leq w_r \leq \infty \text{ where } r = jw_r \tag{2.2}$$

The function G(r) is now written as

$$G(r) = \frac{m_1(r) + n_1(r)}{m_2(r) + n_2(r)} \tag{2.3}$$

where m(r) refers to the terms of the polynomial in even powers of r and n(r) the terms in odd powers of r.

Substituting (2.3) into (2.2) yields

$$\mathrm{Re}\left[\frac{m_1(r) + n_1(r)}{m_2(r) + n_2(r)}\right] + 1/K > 0.$$

$$\mathrm{Re}\left[\frac{K(m_1(r) + n_1(r)) + m_2(r) + n_2(r)}{K(m_2(r) + n_2(r))}\right] > 0 \qquad (2.4)$$

Multiplying (2.4) by $\dfrac{m_2(r) - n_2(r)}{m_2(r) - n_2(r)}$ yields an equation containing many products and sums of m(r) and n(r). The real terms are those containing the products of even terms in r, or those in odd terms of r. Therefore (2.4) reduces to

$$\left[\frac{K(m_1 m_2(r) - n_1 n_2(r)) + m_2^2(r) - n_2^2(r)}{K(m_2^2(r) - n_2^2(r))}\right] > 0.$$

$$K(m_2^2(r) - n_2^2(r)) > 0 \quad \text{for all } w_r$$

Therefore we have

$$m_2^2(r) - n_2^2(r) + K(m_1 m_2(r) - n_1 n_2(r)) > 0.$$

$$\text{Let } x = -r^2 \qquad (2.5)$$

and let $p_1(x) = m_2^2(r) - n_2^2(r) \;\big|\, x = -r^2 = w_r^2$

$$p_2(x) = m_1 m_2(r) - n_1 n_2(r) \;\big|\, x = -r^2 = w_r^2 \qquad (2.6)$$

The stability criterion now becomes

$$p_1(x) + Kp_2(x) > 0 \quad \text{for all } x > 0. \qquad (2.7)$$

The criterion is satisfied if for

$$x \rightarrow \infty \quad p_1(x) + Kp_2(x) \rightarrow \infty, \qquad (2.8)$$

and if there exists a positive value of K such that

$$p_1(x) + Kp_2(x) = 0 \qquad (2.9)$$

has no positive real zeros with the exception of coincident

ones of even multiplicity. This can be easily seen by factoring (2.9) for some K which yields positive real roots.

$$(x + b)(x+d)(x - f + jg)(x - f - jg).....(x - y)^n \qquad (2.10)$$

Equation (2.10) is negative for positive values of x which are less than y when n is odd. When n is even, all values of $x \geq 0$ yield a positive result for (2.10).

Equation (2.9) is easily investigated by use of a root locus diagram with K as the varying parameter. The values of K which yield part of a branch on the positive real axis do not satisfy the criterion. It is definitely not satisfied for any value of K, if a complete branch exists on the positive real axis.

2.2 Examples of the Tsypkin Criterion

(a) $G(s) = \dfrac{1 - e^{-sT}}{s(s + 1)(s + 2)} \qquad T = 1 \text{ sec.}$

$$G(r) = \frac{.5(r - .542r - .458)}{r^2 + 3.47r + 2.84}$$

$$m_1(r) = .5r^2 - .229 \qquad n_1(r) = -.271r$$

$$m_2(r) = r^2 + 2.84 \qquad n_2(r) = 3.47r$$

$p_1(x)$ and $p_2(x)$ are obtained from substituting the above values for m(r) and n(r) into (2.6).

$$p_1(x) = x^2 + 6.41x + 8.12$$

$$p_2(x) = .5(x^2 - 4.246x - 1.34) \qquad (2.11)$$

Substituting (2.11) into (2.9) yields

$$(x + 4.68)(x + 1.73) + .5K(x + .295)(x - 4.54) = 0 \quad (2.12)$$

The root locus plot for (2.12) is shown in Fig. 2.2.

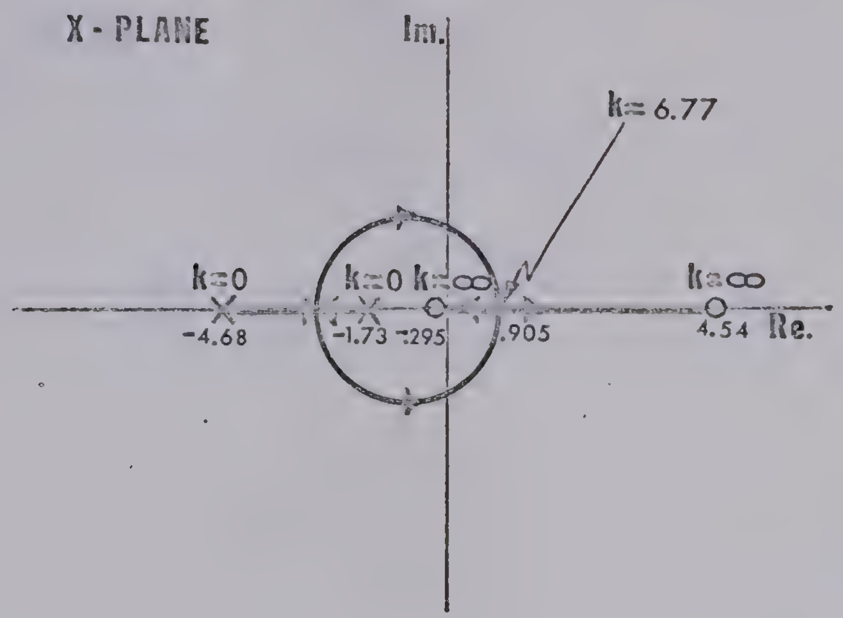

FIGURE 2.2 TSYPKIN ROOT LOCUS FOR $G(s) = \dfrac{1 - e^{-s}}{s(s + 1)(s + 2)}$

K = 6.77 places two roots of (2.12) at x = .905. A further increase in K results in two separate positive roots of (2.12). The criterion is no longer satisfied. Therefore $K < 6.77$.

$$\text{(b)} \quad G(s) = \frac{1 - e^{-sT}}{s(s + 1)} \qquad T = 1 \text{ sec.}$$

$$G(r) = \frac{r - 1}{r + 2.16}$$

$$p_1(x) = x + 4.67$$

$$p_2(x) = x - 2.16 \tag{2.13}$$

Substituting (2.13) into (2.9) yields

$$x + 4.67 + K(x - 2.16) = 0. \tag{2.14}$$

The plot is shown in Fig. 2.3. The maximum allowable value of K occurs at x = 0. Evaluating K at x = 0 produces the result that K = 2.16. Therefore $K < 2.16$.

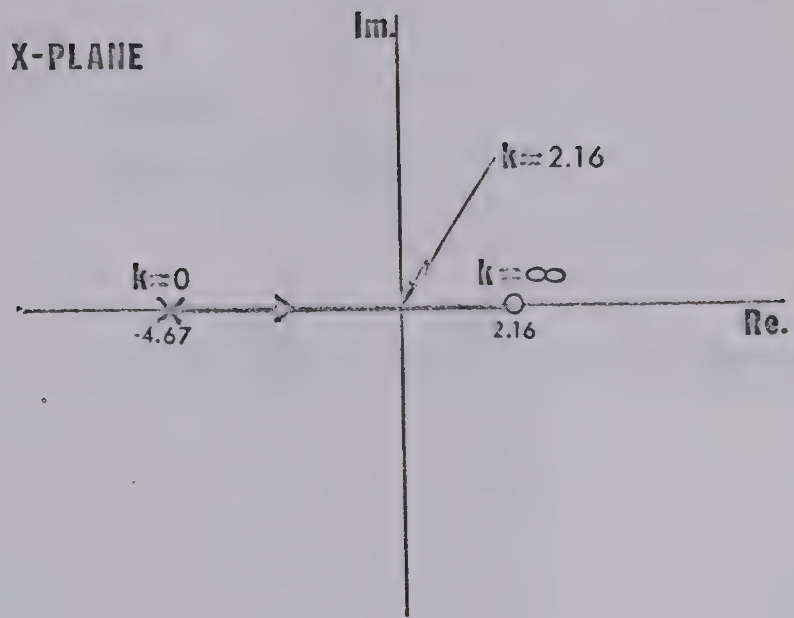

FIGURE 2.3 TSYPKIN ROOT LOCUS FOR $G(s) = \frac{1 - e^{-s}}{s(s + 1)}$

$$\text{(c)} \quad G(s) = \frac{1 - e^{-sT}}{s^2(s + 1)} \qquad T = 1 \text{ sec.}$$

$$G(r) = \frac{.632r^2 - .528r - .104}{1.264r + 2.736}$$

$$p_1(x) = x + 4.67$$

$$p_2(x) = -1.5(x + .119) \qquad (2.15)$$

Substituting (2.15) into (2.9) yields

$$x + 4.67 - 1.5K(x + .119) = 0. \qquad (2.16)$$

Fig. 2.4 is the resulting root locus for (2.16).

A negative sign occurs in (2.16). Applying (2.8) it is seen that as

$x \to \infty \quad x + 4.67 - 1.5K(x + .119) \to x(1 - 1.5K) \to \infty$ for $K < .667$.

From Fig. 2.4 the maximum allowable value of K occurs at $x = \pm\infty$, for which $K = .667$. Therefore $K < .667$. A negative sign will occur in (2.9) for functions which produce

a frequency plot of G(z) entirely within the left hand z-plane.

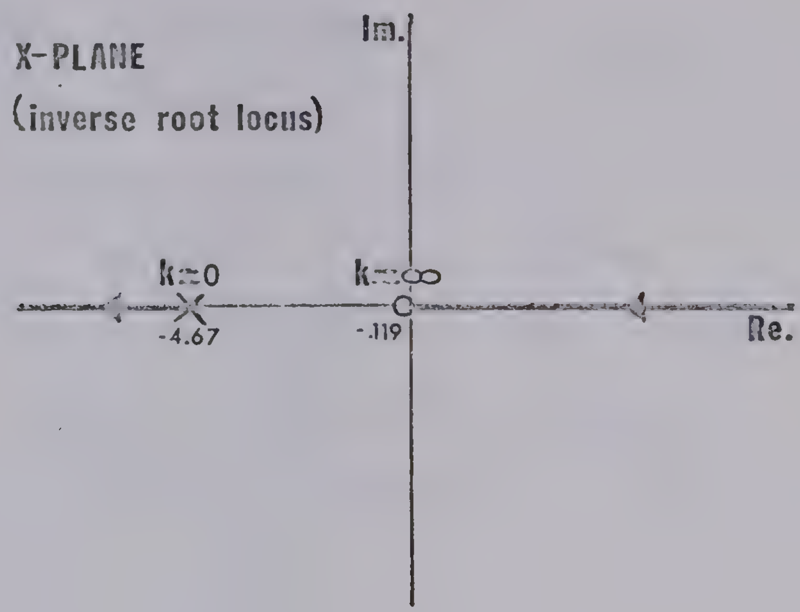

FIGURE 2.4 TSYPKIN ROOT LOCUS FOR $G(s) = \dfrac{1 - e^{-s}}{s^2(s + 1)}$

CHAPTER 3 COMPENSATION BASED ON THE TSYPKIN CRITERION

3.1 Development of the Method

In this chapter a method of filter compensation is developed to achieve improvement in the static accuracy and transient response of a given system.

The Tsypkin criterion in the r-domain is

$$\mathrm{Re}\left[G(r) + 1/K\right] > 0 \quad \text{for } -\infty \leq w_r \leq \infty .$$

A filter will be cascaded with G(r) in the r-domain. When the value of the constants of the filter are known it will be evaluated in the s-domain. The filter will be

$$G_c'(r) = \frac{1 + acr}{1 + cr} \tag{3.1}$$

where a and c are constants to be determined.

$$G(r) = \frac{m_1(r) + n_1(r)}{m_2(r) + n_2(r)}$$

$$G_1(r) = G(r)Gc'(r)$$

$$G_1(r) = \frac{m_1(r) + acrn_1(r) + n_1(r) + acrm_1(r)}{m_2(r) + n_2(r)cr + n_2(r) + m_2(r)cr}$$

The criterion becomes

$$\mathrm{Re}\left[G_1(r) + 1/K\right] > 0 \quad \text{for } -\infty \leq w_r \leq \infty , \tag{3.2}$$

where $G_1(r) = \dfrac{m_1^*(r) + n_1^*(r)}{m_2^*(r) + n_2^*(r)}$ and

$$m_1^*(r) = m_1(r) + acrn_1(r)$$

$$n_1^*(r) = n_1(r) + acrm_1(r)$$

$m_2^*(r) = m_2(r) + n_2(r)cr$ and

$$n_2^*(r) = n_2(r) + m_2(r)cr.$$

Following the same procedure as in section 2.1, the stability criterion now becomes

$$p_1{*}(x) + Kp_2{*}(x) > 0 \quad \text{for all } x \geq 0. \tag{3.3}$$

The criterion is satisfied if for

$$x \to \infty \quad p_1{*}(x) + Kp_2{*}(x) \to \infty, \tag{3.4}$$

and if there exists a positive value of K such that

$$p_1{*}(x) + Kp_2{*}(x) = 0 \tag{3.5}$$

has no positive real zeros with the exception of coincident ones of even multiplicity.

$$\begin{aligned} p_1{*}(x) &= m_2^2{*}(r) - n_2^2{*}(r) \Big|_{x = -r^2} \\ &= m_2^2(r) - n_2^2(r) - c^2r^2(m_2^2(r) - n_2^2(r)) \Big|_{x = -r^2} \end{aligned}$$

From (2.6) we obtain

$$p_1{*}(x) = c^2 p_1(x)(x + 1/c^2).$$

$$\begin{aligned} p_2{*}(x) &= m_1{*}m_2{*}(r) - n_1{*}n_2{*}(r) \Big|_{x = -r^2} \\ &= m_1m_2(r) - n_1n_2(r) + cr(m_1n_2(r) - m_2n_1(r)) \\ &\quad + ac(cx(m_1m_2(r) - n_1n_2(r)) - r(m_1n_2(r) - n_1m_2(r)) \Big|_{x = -r^2} \end{aligned}$$

Let $p_3(x) = r(m_1n_2(r) - n_1m_2(r)) \Big|_{x = -r^2}$.

With (2.6) and the above equation we have

$$p_2{*}(x) = p_2(x) + cp_3(x) + ac^2xp_2(x) - p_3(x)ac.$$

(3.5) now becomes

$$\begin{aligned} &(x + 1/c^2)p_1(x) + K/c^2(p_2(x) + cp_3(x)) \\ &\qquad + aK(xp_2(x) - p_3(x)/c) = 0 \end{aligned} \tag{3.6}$$

To obtain the required filter it will be necessary to

select values of c and evaluate K and a from (3.6) by the root locus method.

3.2 Examples of the Compensation Technique

$$\text{(a)} \quad G(s) = \frac{1 - e^{-sT}}{s^2(s + 1)} \qquad T = 1 \text{ sec.}$$

$$p_3(x) = .5x^2 - .8225x \tag{3.7}$$

Substituting (2.15) and (3.7) into (3.6) yields

$$(x + 1/c^2)(x + 4.67) + K/c^2(-1.5x - .178 + c(.5x^2 - .8225x)) + aK(x(-1.5x - .178) - 1/c(.5x^2 - .8225x)) = 0 \tag{3.8}$$

i) Choose c = .5

Substituting c into (3.8) yields

$$(x + 4)(x + 4.67) + K(x - 7.732)(x + .092) - 2.5aKx(x - .568) = 0 \tag{3.9}$$

For a = 0 the path of the poles of (3.9) is plotted in Fig. 3.1.

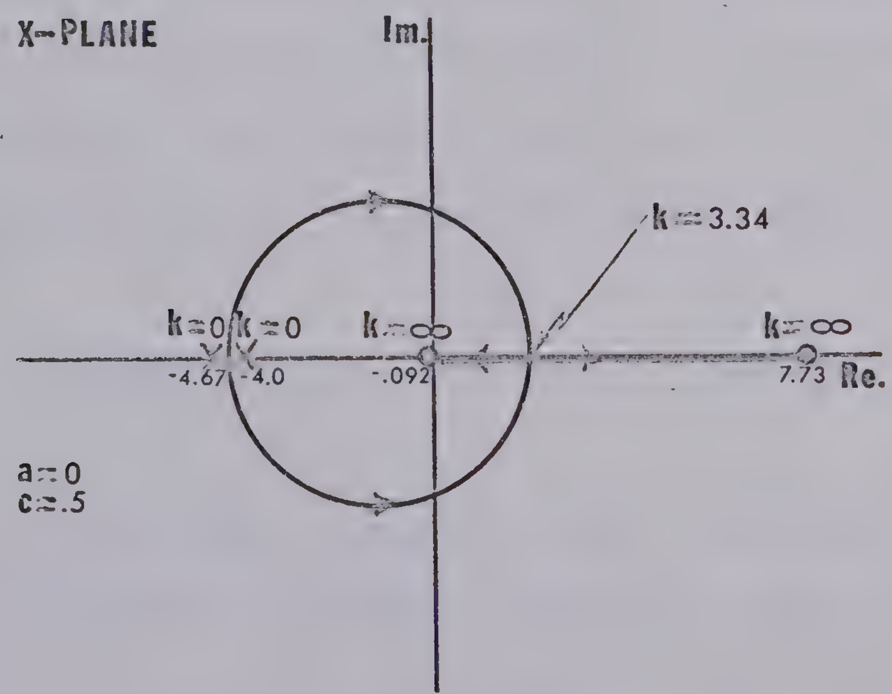

FIGURE 3.1 COMPENSATION FOR $G(s) = \dfrac{1 - e^{-s}}{s^2(s + 1)}$, c = .5, PART A

Fig. 3.2 is the root locus plot of (3.9) with $a \neq 0$.

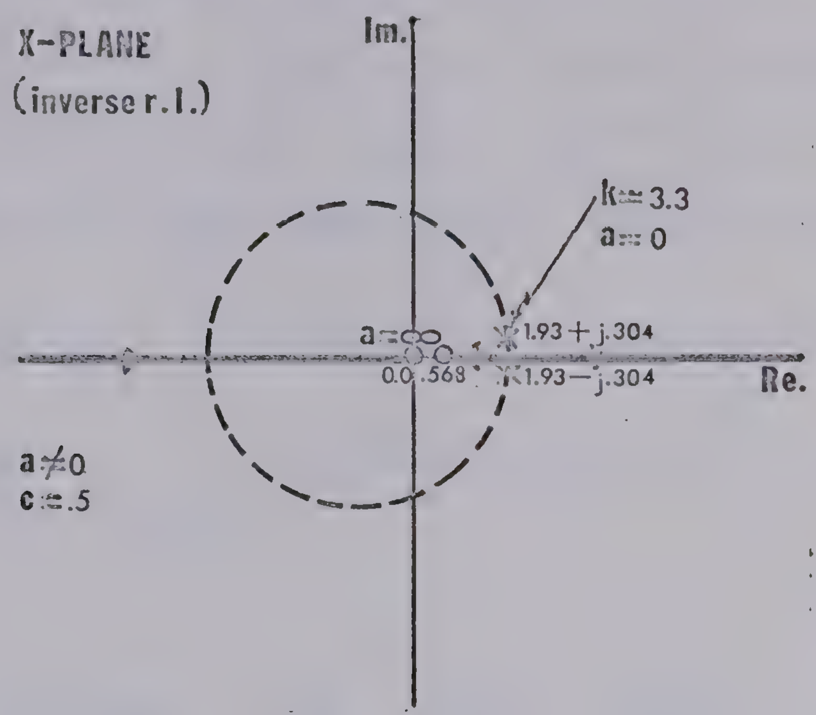

FIGURE 3.2 COMPENSATION FOR $G(s) = \dfrac{1 - e^{-s}}{s^2(s + 1)}$, c = .5, PART B

From Fig. 3.2 it can be seen that if K is made equal to 3.34, a must equal zero to satisfy the criterion. This will result in an unrealizable filter. One way of circumventing this problem is to choose a value of $K < 3.34$, and solve for the resulting a which will produce two coincident roots of (3.9) on the positive real axis. In this case let K = 3.3. (3.9) becomes

$$(x - 1.93 - j.304)(x - 1.93 + j.304) - 1.92ax(x - .568) = 0. \quad (3.10)$$

From Fig. 3.2 $a < .0201$. Let a = .02. The filter in the r-domain is obtained by substituting for a and c into (3.1).

$$G_c'(r) = \frac{1 + .01r}{1 + .5r}$$

$$G(r) = \frac{.5r^2 - .418r - .0825}{r + 2.16}$$

$$G_c'(r)G(r) = \frac{(r - 1)(r + .165)(.01)(r + 100)}{(r + 2)(r + 2.16)}$$

In order to obtain the filter in the s-domain a convenient method by Kuo[3] will be used. Let $G'(s)$ equal the plant without the zero order hold. For this case

$G'(s) = \frac{1}{s(s + 1)}$. Therefore the z-transform of

$$G_c(s)G'(s)/s \Big|_{z = \frac{r+1}{r-1}} = \frac{.01(r - 1)(r + 1)}{2}\left[1 + \frac{1218}{r + 2.16} - \frac{1112}{r + 2}\right].$$

$$G_c(s)G'(s) = \frac{.02}{s} + \frac{12.18}{s + 1} - \frac{12.35}{s + 1.1}$$

$$G_c(s) = \frac{-.15(s - 7.29)(s + .02)}{s + 1.1}$$

To make the filter realizable a remote pole must be added to $G(s)$. For instance add $100/(s + 100)$.

Therefore the final filter is $G_c(s) = \frac{-15(s - 7.29)(s + .02)}{(s + 1.1)(s + 100)}$

ii) Choose $c = .99$

Substituting c into (3.8) produces

$$(x + 4.67)(x + 1.02) + .505K(x - 4.75)(x + .076) - 2.01aKx(x - .326) = 0. \quad (3.11)$$

For $a = 0$ the path of the poles of 3.11 is plotted in Fig. 3.3. Fig. 3.4 is the root locus plot of (3.11) with $a \neq 0$. From Fig. 3.4 it can be seen that if $K = 5.65$, a must equal zero. Therefore choose K less than 5.65. For this case let $K = 2.75$. (3.11) becomes

$$(x - .169 - j1.33)(x - .169 + j1.33) - 2.31ax(x - .326) = 0. \quad (3.12)$$

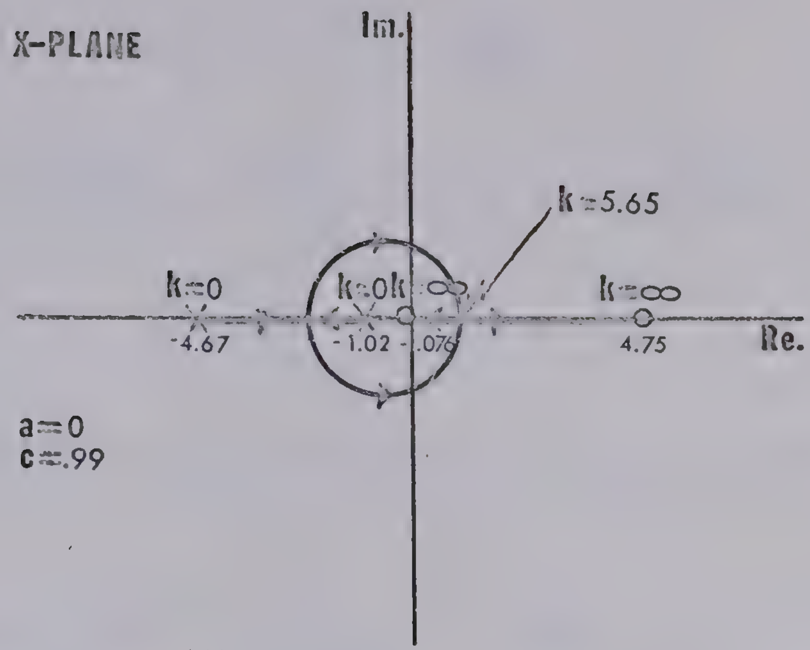

FIGURE 3.3 COMPENSATION FOR $G(s) = \dfrac{1 - e^{-s}}{s^2(s + 1)}$, c = .99, A

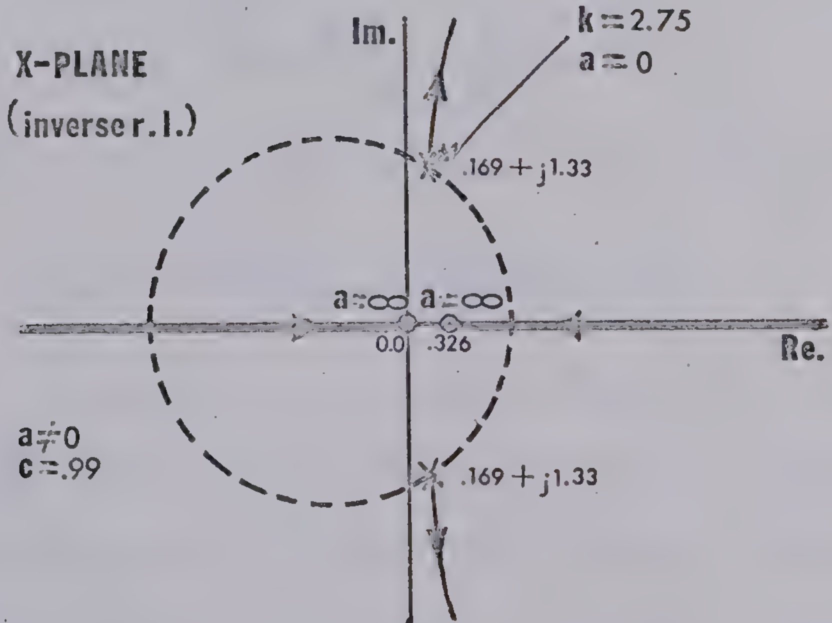

FIGURE 3.4 COMPENSATION FOR $G(s) = \dfrac{1 - e^{-s}}{s^2(s + 1)}$, c = .99, B

For large x (3.12) approaches (1 - 2.31a)x. Applying (3.4) we see that as $x \to \infty$ $(1 - 2.31a)x \to \infty$ only for $a < .433$. The same result, $a < .433$, is obtained from the root locus plot of (3.12). Choose a = .43.

$$G_c'(r) = \frac{1 + .425r}{1 + .990r}$$

$$\frac{G_c'(r)G(r)}{r - 1} = \frac{.215(r + .165)(r + 2.35)}{(r + 1.01)(r + 2.16)}$$

$$G'(s) = \frac{1}{s(s + 1)}$$

The z-transform of $G_c(s)G'(s)/s \Big|_{z = \frac{r+1}{r-1}}$ equals

$$\frac{.215(r - 1)(r + 1)}{2} \; 1 + \frac{.332}{r + 2.16} - \frac{.983}{r + 1.01} \; .$$

$$\frac{G_c(s)G'(s)}{s} = \frac{.43}{s^2} + \frac{.0715}{s(s + 1)} - \frac{1.12}{s(s + 5.3)}$$

$$G_c(s) = \frac{-.62(s - 4.08)(s + .9)}{s + 5.3}$$

Again with the addition of a remote pole the final $G_c(s)$ is

$$G_c(s) = \frac{-62(s - 4.08)(s + .9)}{(s + 5.3)(s + 100)}$$

In these examples the addition of filters has certainly improved the gain sector from the value of .667 obtained from section 2.2c for this system. The unknown result is the transient response of the new systems. In many examples it was found that with the addition of a filter the transient response becomes extremely sluggish. It was found not possible to predict in what way different systems would be affected.

CHAPTER 4 JURY AND LEE CRITERION NO. 1

4.1 Development of the Root Locus Method of Evaluation

For systems of the type shown in Fig. 1.1 and non-linear elements with the properties and form as depicted in Fig. 4.1, stability is guaranteed if[4]

$$\mathrm{Re}\left[G(z)\left(1 + \frac{q(z-1)}{z}\right)\right] + 1/K - \frac{K'q\left|(z-1)G(z)\right|^2}{2} \geq 0$$

for $|z| = 1$ and some $q \geq 0$. (4.1)

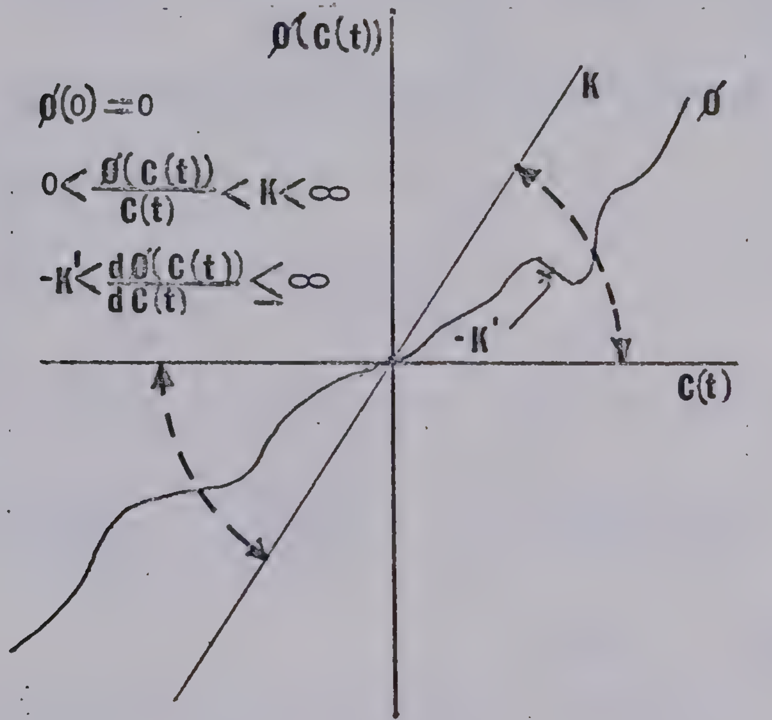

FIGURE 4.1 JURY AND LEE CRITERION NO. 1

By use of the bilinear transform (4.1) becomes

$$\mathrm{Re}\left[G(r)\left(1 + \frac{2q}{r+1}\right)\right] + 1/K - \frac{K'q\left|\frac{2G(r)}{r-1}\right|^2}{2} \geq 0 \qquad (4.2)$$

for $-\infty \leq w_r \leq \infty$ where $r = jw_r$.

$$G(r) = \frac{m_1(r) + n_1(r)}{m_2(r) + n_2(r)}$$

$$\mathrm{Re}\ G(r) = \frac{m_1m_2(r) - n_1n_2(r)}{m_2(r)^2 - n_2(r)^2}$$

$$\mathrm{Re}\left[\frac{G(r)q2}{r+1}\right] = \frac{2q\left[m_1m_2(r) - n_1n_2(r) + r(m_1n_2(r) - n_1m_2(r))\right]}{m_2^2(r) - n_2^2(r) - r^2(m_2^2(r) - n_2^2(r))}$$

$$\mathrm{Re}\left[\frac{G(r)}{r-1}\right] = \frac{-m_1m_2(r) + rm_1n_2(r) + n_1n_2(r) - rm_2n_1(r)}{(1-r^2)(m_2^2(r) - n_2^2(r))}$$

$$j\mathrm{Im}\left[\frac{G(r)}{r-1}\right] = \frac{-r(m_1m_2(r) - n_1n_2(r)) + m_1n_2(r) - m_2n_1(r)}{(1-r^2)(m_2^2 - n_2^2)}$$

$$\left|\frac{G(r)}{r-1}\right|^2 = \left(\mathrm{Re}\left[\frac{G(r)}{r-1}\right]\right)^2 - \left(j\mathrm{Im}\left[\frac{G(r)}{r-1}\right]\right)^2$$

Substituting (2.5) and (2.6) and the following

$$p_3(x) = r(m_1n_2(r) - n_1m_2(r))\Big|_{x=-r^2}$$

$$p_4(x) = m_1^2(r) - n_1^2(r)\Big|_{x=-r^2} \qquad (4.3)$$

into the above equations and then into (4.2) we obtain

$$\frac{p_2(x)}{p_1(x)} + \frac{1}{K} + \frac{2q(p_3(x) + p_2(x))}{(1+x)p_1(x)} - \frac{2K'qp_4(x)}{(1+x)p_1(x)} \geq 0$$

for all $x \geq 0$.

Now $p_1(x) \geq 0$ for all $x \geq 0$, and $K \geq 0$. Therefore we obtain

$$p_1(x) + Kp_2(x) + \frac{2qK(p_3(x) + p_2(x))}{x+1} - \frac{2KK'qp_4(x)}{x+1} \geq 0$$

for all $x \geq 0$. (4.4)

(4.4) must satisfy the same conditions as (2.7). (4.4) is investigated by use of the root locus for varying K and q.

4.2 Examples of Jury and Lee No. 1

(a) $G(s) = \dfrac{1 - e^{-sT}}{s(s+1)(s+2)}$ $\quad T = 1$ sec.

For this system

$$p_3(x) = 2.008x^2 + .055x \quad \text{and}$$

$$p_4(x) = .25x^2 + .3068x + .0552. \qquad (4.5)$$

Substituting (2.11) and (4.5) into (4.4) and setting the result equal to zero yields

$$(x + 4.68)(x + 1.73) + .5K(x - 4.54)(x + .295)$$

$$+ \frac{2qK(2.508)(x - 1.074)(x + .248)}{x + 1}$$

$$- \frac{2qKK'(.25x^2 + .3068x + .0552)}{x + 1} = 0. \quad (4.6)$$

K' is chosen as the value of the largest negative a. c. gain (slope of curve at any point) of the nonlinearity and the resulting K for this value is determined from (4.6). As an example, K' = 0 would fit a dead zone type nonlinearity. With K' = 0 (4.6) reduces to

$$(x + 4.68)(x + 1.73) + .5K(x - 4.54)(x + .295)$$

$$+ \frac{2qK(2.508)(x - 1.074)(x + .248)}{x + 1} = 0. \qquad (4.7)$$

For q = 0 the path of the poles of (4.7) is already plotted (Fig. 2.2). As soon as $K > 6.77$ in Fig. 4.2, there are two roots on the positive real axis and no value of q will remove both of them. Therefore, for K' = 0, $K \leq 6.77$ and q = 0.

For other values of K' the same value of K is found. The sector cannot be increased, and there is no improvement over the use of the Tsypkin theorem for this Jury and Lee criterion.

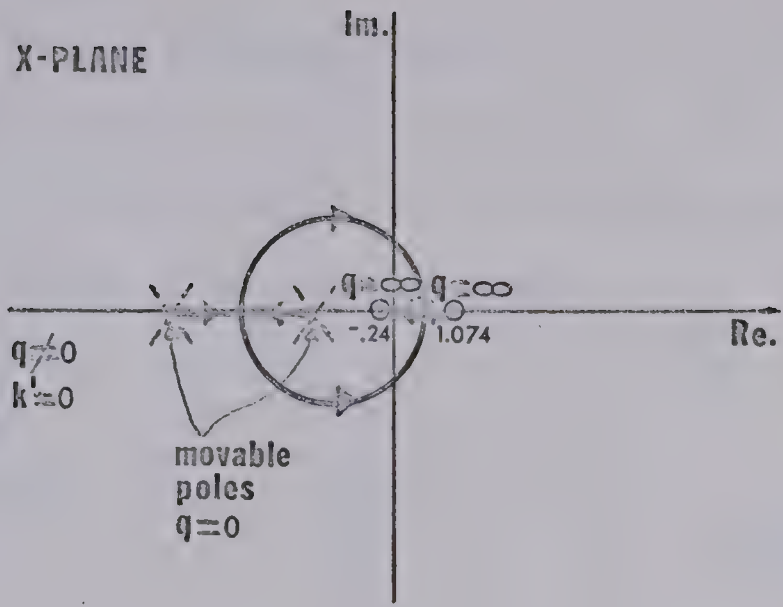

FIGURE 4.2 J. AND L. NO. 1 ROOT LOCUS $G(s) = \dfrac{1 - e^{-s}}{s(s + 1)(s + 2)}$

$$\text{(b)} \quad G(s) = \frac{1 - e^{-sT}}{s^2(s + 1)} \qquad T = 1 \text{ sec.}$$

$$p_3(x) = .5x^2 - .8225x$$

$$p_4(x) = .25x^2 + .2575x + .0064 \qquad (4.8)$$

Substituting (2.15) and (4.8) into (4.4) and setting the result equal to zero yields

$$(x + 4.67) - 1.5K(x + .119) + \frac{2qK(.5)(x - 4.725)(x + .075)}{x + 1}$$

$$- \frac{2KK'q(.25x^2 + .2575x + .0064)}{x + 1} = 0. \quad (4.9)$$

i) $K' = 0$

(4.9) reduces to

$$(x + 4.67) - 1.5K(x + .119)$$

$$+ \frac{2qK(.5)(x - 4.725)(x + .075)}{x + 1} = 0. \qquad (4.10)$$

For q = 0, Fig. 2.4 becomes the root locus plot of the poles of (4.10). From Fig. 4.3 it can be seen that it is possible to find a q such that there are no positive real roots of (4.10) for a given K, up to x = 4.725.

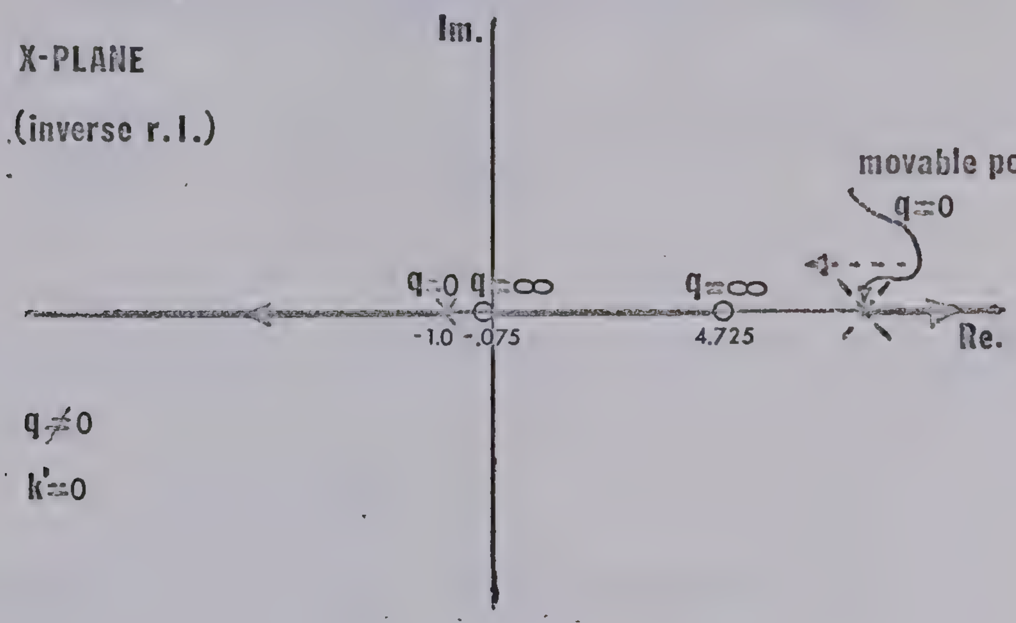

FIGURE 4.3 J. AND L. NO. 1 ROOT LOCUS $G(s) = \dfrac{1 - e^{-s}}{s^2(s + 1)}$, A

For values of K which yield the position of the movable pole at less than 4.725, no q can be found. The value of K at x = 4.725 is

$$K = \left.\frac{x + 4.67}{1.5(x + .119)}\right|_{x = 4.725} = 1.3.$$

Substituting K = 1.3 into (4.10) results in

$$-.95(x - 4.725) + \frac{1.3q(x - 4.725)(x + .075)}{x + 1} = 0. \qquad (4.11)$$

From Fig. 4.4 it can be seen that a branch of the root locus of (4.11) will pass through x = 4.725. If q is evaluated at this point a double root of (4.10) will be produced, satisfying the criterion. The value of q at x = 4.725 is

.87. Therefore $K \leq 1.3$, for $K' = 0$ and $q = .87$.

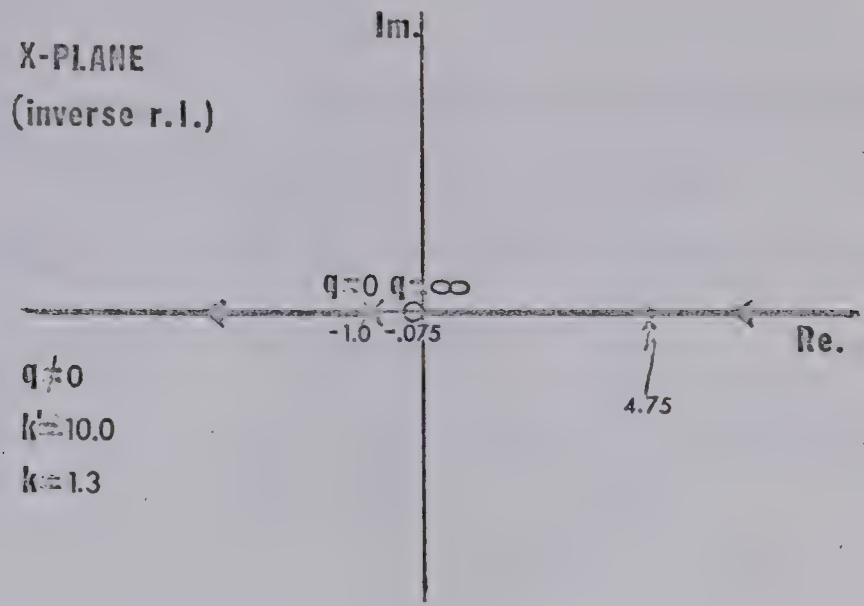

FIGURE 4.4 J. AND L. NO. 1 ROOT LOCUS $G(s) = \dfrac{1 - e^{-s}}{s^2(s + 1)}$, B

ii) $K' = 1$

Substituting $K' = 1$ into (4.9) produces

$$(x + 4.67) - 1.5K(x + .119) + \frac{.5qK(x - 10.39)(x + .07)}{x + 1} = 0. \qquad (4.12)$$

The same evaluation procedure as for $K' = 0$ in part i holds for this case. The result is $K \leq .965$, for $K' = 1$ and a q of .988.

iii) $K' = 2$

Substituting $K' = 2$ into (4.9) produces

$$(x + 4.67) - 1.5K(x + .119) - \frac{5.68qK(x + .08)}{x + 1} = 0. \qquad (4.13)$$

As $x \rightarrow \infty$ (4.13) approaches ∞ only for $K < .667$. The root locus investigation of (4.13) yields the same result. This is the Tsypkin value. This is the same result as is obtained for all $K' \geq 2$. Therefore for this example the Jury and Lee criterion reduces to the Tsypkin criterion for $K' \geq 2$.

CHAPTER 5 JURY AND LEE CRITERION NO. 2

5.1 Development of the Root Locus Method of Evaluation

For systems of the type shown in Fig. 1.1 and non-linear elements with the properties and form as shown in Fig. 5.1, stability is guaranteed if[5]

$$\mathrm{Re}\left[G(z)(1 + q(z - 1))\right] + \frac{1}{K} - \frac{|q| K' \left|(z - 1)G(z)\right|^2}{2} \geq 0$$

$$\text{for } |z| = 1 \text{ and some } q. \qquad (5.1)$$

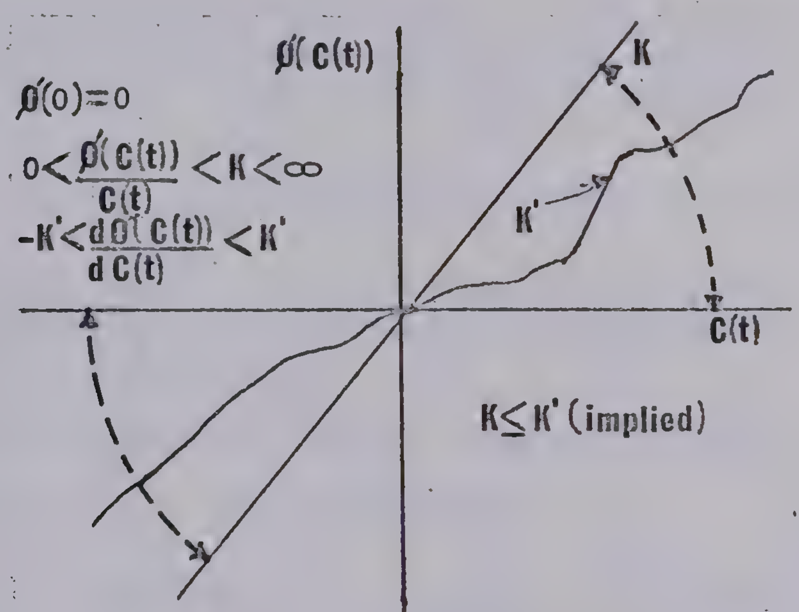

FIGURE 5.1 JURY AND LEE CRITERION NO. 2

Jury and Lee, in one of their papers[4], state that they have not found practical cases where $q < 0$ produced meaningful results. Therefore only $q \geq 0$ will be investigated. The criterion reduces to

$$\phi(0) = 0$$

$$0 < \frac{\phi(C(t))}{C(t)} < K < \infty$$

$$-\infty \leq \frac{d\phi(C(t))}{dC(t)} < K'$$

$$\mathrm{Re}\left[G(z)(1 + q(z - 1))\right] + \frac{1}{K} - \frac{qK'\left|(z - 1)G(z)\right|^2}{2} \geq 0$$

$$\text{for } |z| = 1 \text{ and } q \geq 0. \qquad (5.2)$$

By use of the bilinear transform (5.2) becomes

$$\mathrm{Re}\left[G(r)\left(1 + \frac{2q}{r - 1}\right)\right] + \frac{1}{K} - \frac{K'q\left|\frac{2G(r)}{r - 1}\right|^2}{2} \geq 0$$

$$\text{for } -\infty \leq w_r \leq \infty \text{ and } q \geq 0. \qquad (5.3)$$

$$G(r) = \frac{m_1(r) + n_1(r)}{m_2(r) + n_2(r)}$$

$$\mathrm{Re}\left[\frac{2qG(r)}{r - 1}\right] = \frac{2q(-m_1m_2(r) + n_1n_2(r) + r(m_1n_2(r) - m_2n_1(r)))}{(1 - r^2)(m_2^2(r) - n_2^2(r))}$$

Using (4.3), (2.5), and (2.6) the above equation becomes

$$\mathrm{Re}\left[\frac{2qG(r)}{r - 1}\right]_{x = -r^2} = \frac{2q(p_3(x) - p_2(x))}{(x + 1)p_1(x)} .$$

With the above equation, (2.5), and the other substitutions used in section 4.1, (5.3) becomes

$$p_1(x) + Kp_2(x) + \frac{2qK(p_3(x) - p_2(x))}{x + 1} - \frac{2KK'qp_4(x)}{x + 1} \geq 0$$

$$\text{for all } x \geq 0 \text{ and some } q \geq 0. \qquad (5.4)$$

(5.4) must satisfy the same conditions as (2.7).

From Fig. 5.1 it can be seen that to completely fill the sector $K \leq K'$. If $K > K'$ the sector is limited to K' and if $K < K'$ the sector is limited to K. If $K = K'$ the maximum sector width is obtained. In the evaluation of this criterion by the root locus method a value of K' is picked (Tsypkin value), and K and q are evaluated. If $K > K'$ the value of K' is too low and a larger value is

picked. If K = K' = Tsypkin value the sector cannot be improved. If the new value of K is less than K' then K' is too large. This continues until a value of K' = K is found.

5.2 Examples of the Jury and Lee Criterion No. 2

$$\text{(a)} \quad G(s) = \frac{1 - e^{-sT}}{s(s+1)(s+2)} \qquad T = 1 \text{ sec.}$$

Substituting (2.11) and (4.5) into (5.4) and setting the result equal to zero yields

$$(x + 4.68)(x + 1.73) + .5K(x - 4.54)(x + .295) + 2qK(1.508x + .67) - 2KK'q(.25x + .056) = 0. \quad (5.5)$$

For this example K (Tsypkin) = 6.77.

i) Try K' = 6.77.

Substituting the above into (5.5) yields

$$(x + 4.68)(x + 1.73) + .5K(x - 4.54)(x + .295) - .364qK(x - 1.59) = 0. \quad (5.6)$$

For q = 0 the path of the poles of (5.6) for varying K is plotted in Fig. 2.2. It is possible to find a q such that (5.6) has no positive real roots up to the value of K which places a pole at x = 1.59 in Fig. 5.2.

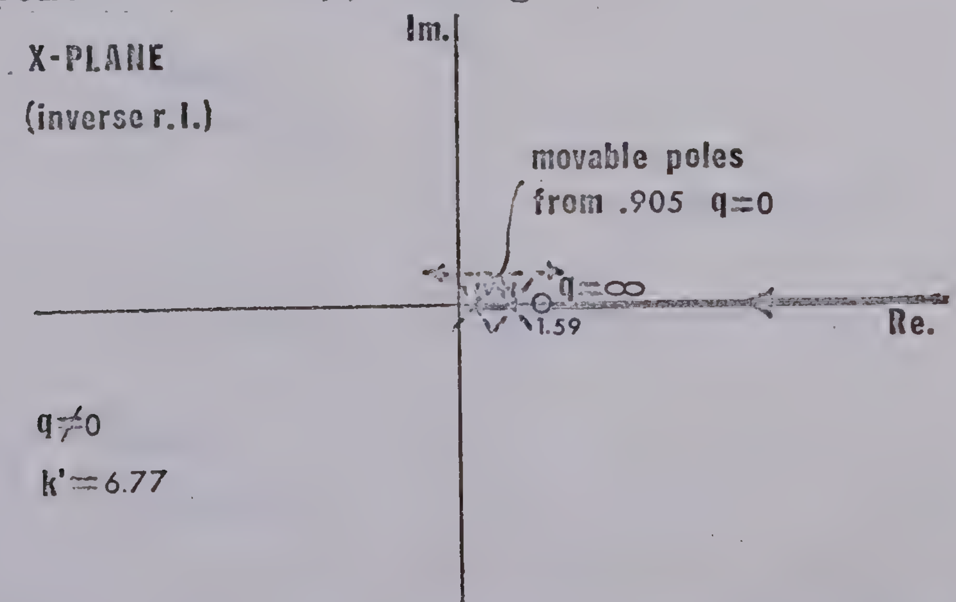

FIGURE 5.2 J. AND L. NO. 2 ROOT LOCUS $G(s)=\frac{1-e^{-s}}{s(s+1)(s+2)}$, A

Evaluating K at $x = 1.59$ produces $K \doteq 7.48$. Therefore $K \leq 7.48$. This produces one root of (5.6) at $x = 1.59$. But the roots must be of even multiplicity, so q must be evaluated at 1.59 also. However, since $K > K'$, K' was chosen too low for the maximum sector.

ii) Try $K' = 7.00$

This value of K' produces a $K \leq 6.85$, which means that $K' > K$. Therefore K' is too large.

iii) Try $K' = 6.95$

Substituting this into (5.5) produces

$$(x + 4.68)(x + 1.73) + .5K(x - 4.54)(x + .295)$$
$$- .460qK(x - 1.23) = 0. \quad (5.7)$$

Evaluating K at $x = 1.23$ yields $K \leq 6.95$. This will give the maximum sector, and q must now be evaluated at $x = 1.23$ to obtain two roots of (5.7) at this point. Substituting $K = 6.95$ into (5.7) yields

$$4.475(x - 1.23)(x - .635) - 3.195q(x - 1.23) = 0. \quad (5.8)$$

The root locus diagram of (5.8) is shown in Fig. 5.3. Evaluating q at $x = 1.23$ results in $q = .83$.

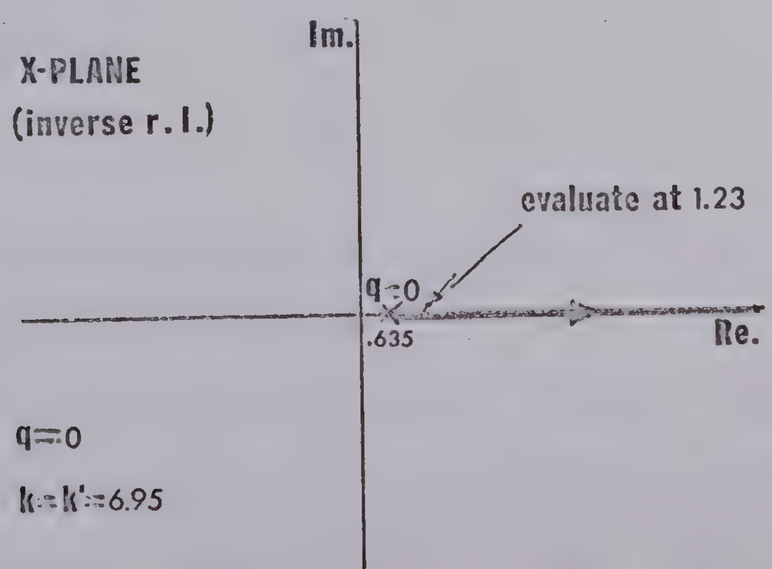

FIGURE 5.3 J. AND L. NO. 2 ROOT LOCUS $G(s)=\frac{1-e^{-s}}{s(s+1)(s+2)}$, B

(b) $G(s) = \dfrac{1 - e^{-sT}}{s^2(s + 1)}$ $T = 1$ sec.

Substituting (2.15) and (4.8) into (5.4) and setting the result equal to zero produces

$$(x + 4.67) - 1.5K(x + .119) + 2qK(.5x + .178)$$
$$-2KK'q(.25x + .0068) = 0. \qquad (5.9)$$

i) Let $K' = .667$ (Tsypkin value)

(5.9) becomes

$$(x + 4.67) - 1.5K(x + .119) + 2qK(.333)(x + 5.2) = 0. \; (5.10)$$

Fig. 2.4 is the root locus plot of the poles of (5.10). From Fig. 5.4 it can be seen that K can take on any value and a q can still be found so that there are no positive real roots of (5.10). Therefore K' is too low.

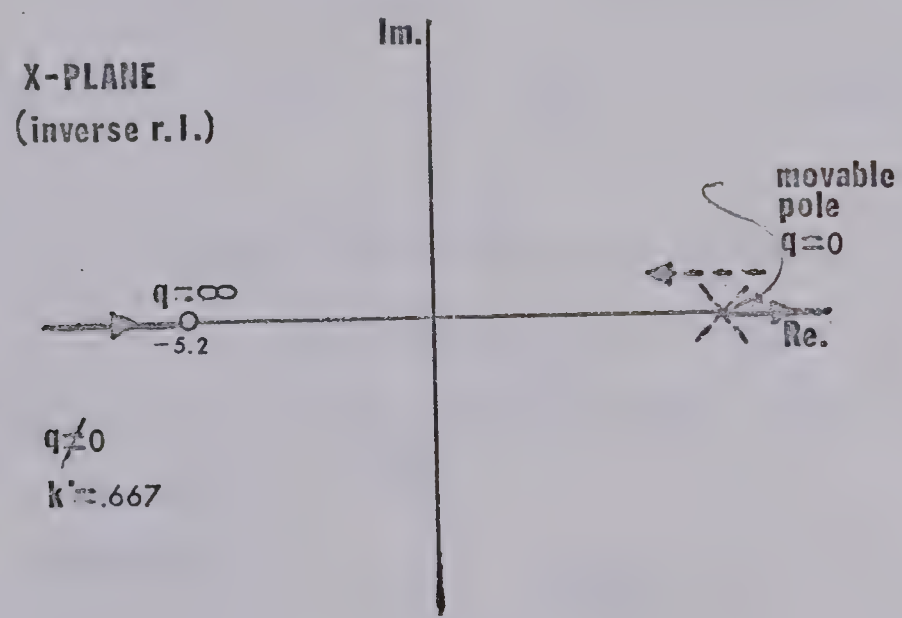

FIGURE 5.4 J. AND L. NO. 2 ROOT LOCUS $G(s) = \dfrac{1 - e^{-s}}{s^2(s + 1)}$

The same result is obtained for all $K' \leq 2$. For $K' > 2$, (5.9) does not approach infinity as x approaches infinity for $K > .667$. Therefore, for $K' = 2$, $K = \infty$. But this means the nonlinearity will never exceed a sector width of K'.

Therefore, $K \leq 2$, for $K' = 2$.

$$\text{(c)} \quad G(s) = \frac{(1 - e^{-sT})(s + 1)}{s^2(s + 2)} \qquad T = 1 \text{ sec.}$$

$$G(r) = \frac{.216r^2 + .284r - .5}{1.135 + .865r}$$

$$m_1(r) = .216r^2 - .5 \qquad n_1(r) = .284r$$

$$m_2(r) = 1.135 \qquad n_2(r) = .865r$$

Substituting m(r) and n(r) into (2.6) and (4.3) yields

$$p_1(x) = .75(x + 1.72)$$

$$p_2(x) = -.568$$

$$p_3(x) = .187x^2 + .755x$$

$$p_4(x) = .0467(x + 5.35)(x + 1). \qquad (5.11)$$

Substituting (5.11) into (5.4) which is set equal to zero yields

$$.75(x + 1.72) - .568K + 2qK(.187)(x + 3.04)$$
$$- 2KK'q(.0467)(x + 5.35) = 0. \qquad (5.12)$$

The root locus for q = 0 is shown in Fig. 5.5.

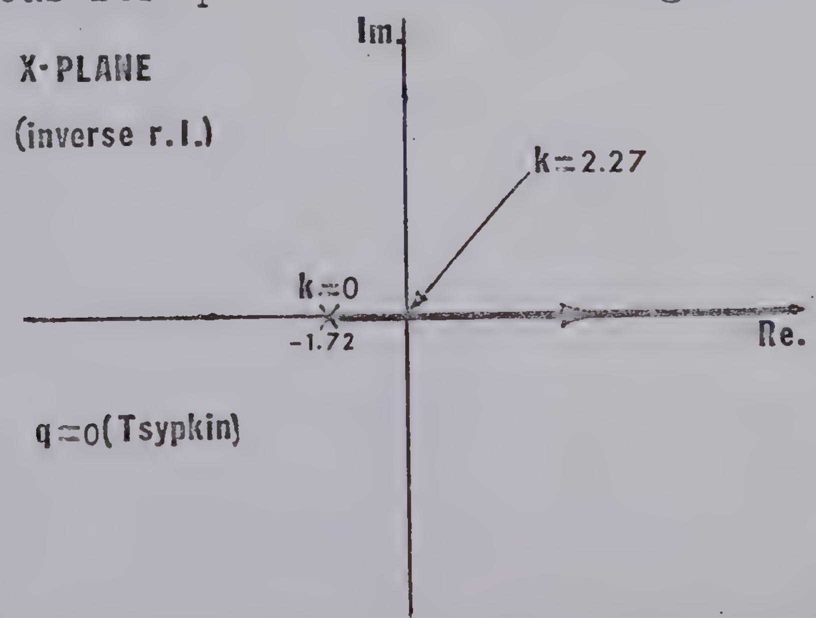

FIGURE 5.5 TSYPKIN ROOT LOCUS $G(s) = \frac{(1 - e^{-s})(s + 1)}{s^2(s + 2)}$

The Tsypkin value from Fig. 5.5 is $K < 2.27$.

i) Try $K' = 2.27$

(5.12) becomes

$$.75(x + 1.72) - K(.568) + 2qK(.081)x = 0 \qquad (5.13)$$

The root locus for (5.13) is plotted in Fig. 5.6

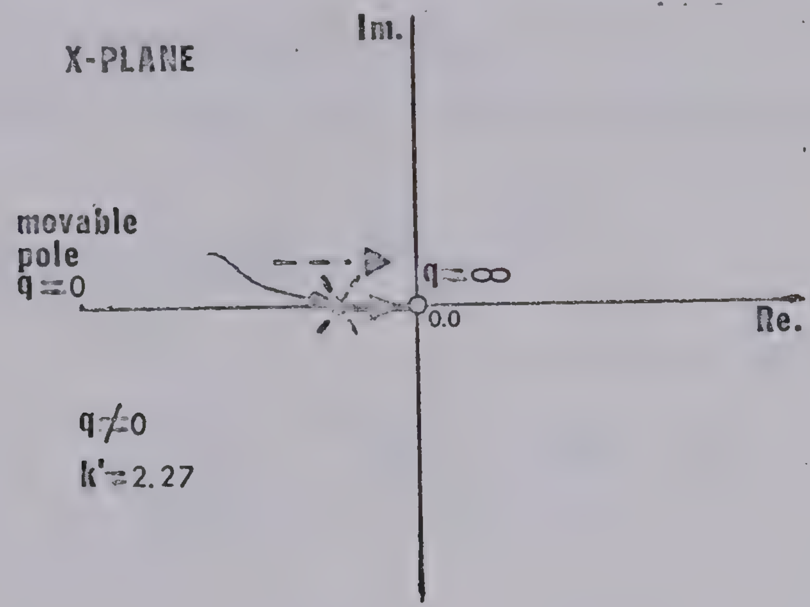

FIGURE 5.6 J. AND L. NO. 2 ROOT LOCUS $G(s) = \dfrac{(1 - e^{-s})(s + 1)}{s^2(s + 2)}$

From Fig. 5.6 it can be seen that K cannot be larger than the value at $x = 0$, or the criterion will be violated. Therefore $K \leq 2.27$ for $K' = 2.27$. The Tsypkin and Jury and Lee criteria produce the same sector width for this system.

CHAPTER 6 JURY AND LEE CRITERION NO. 3

6.1 Development of the Root Locus Method of Evaluation

This criterion deals with monotonically increasing nonlinearities. It is valid for systems of the type shown in Fig. 1.1 and nonlinear elements with the properties and form as shown in Fig. 6.1. Stability is guaranteed if[6]

$$\mathrm{Re}\left[G(z)(1 + qQ(z))\right] + \frac{1}{K} + \frac{|q||z - 1|^2}{2K'} > 0$$

$$\text{for } |z| = 1 \text{ and some } q. \quad (6.1)$$

$$Q(z) = \frac{z - 1}{z} \text{ if } q \geq 0 \text{ or}$$

$$Q(z) = z - 1 \text{ if } q > 0.$$

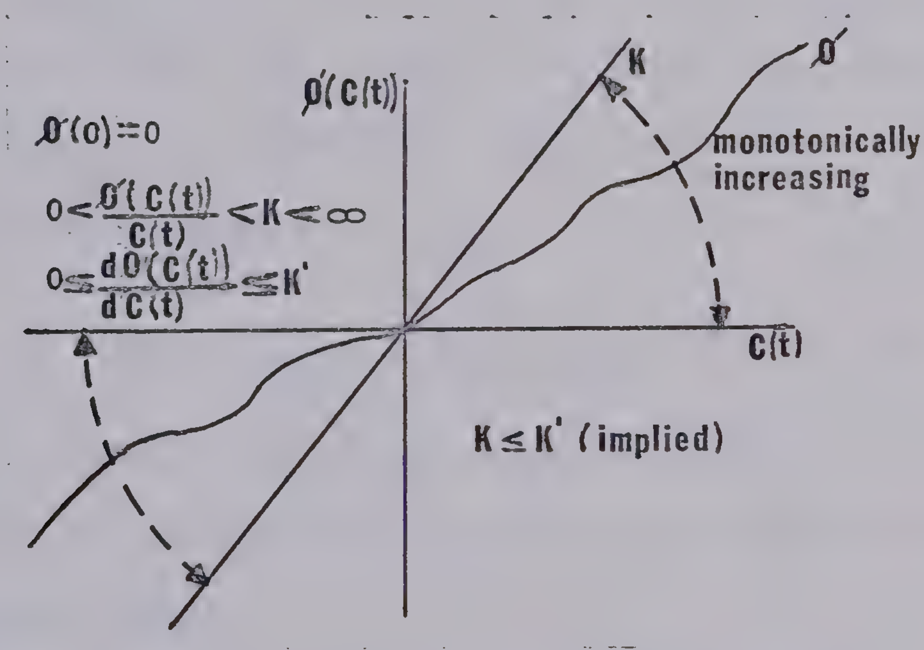

FIGURE 6.1 JURY AND LEE CRITERION NO. 3

By use of the bilinear transformation (6.1) becomes

$$\mathrm{Re}(G(r)(1 + qQ(r)) + \frac{1}{K} + \frac{|q|\left|\frac{2}{r - 1}\right|^2}{2K'} > 0$$

$$\text{for } -\infty \leq w_r \leq \infty \text{ and some } q. \quad (6.2)$$

$Q(r)$ equals $\frac{2}{r + 1}$ for $q \geq 0$ and $\frac{2}{r - 1}$ for $q < 0$.

Once again making the same substitutions used in previous sections the criterion becomes

$$p_1(x) + Kp_2(x) + \frac{2qK(p_3(x) + p_2(x))}{x + 1} + \frac{2qKp_1(x)}{K'(x + 1)} > 0$$

for all $x \geq 0$ and $q \geq 0$ and (6.3)

$$p_1(x) + Kp_2(x) + \frac{2qK(p_2(x) - p_3(x))}{x + 1} + \frac{2qKp_1(x)}{K'(x + 1)} > 0$$

for all $x \geq 0$ and $q < 0$. (6.4)

(6.4) has the negative sign of q already included. (6.3) and (6.4) must satisfy the same conditions as (2.7).

6.2 Examples of the Jury and Lee Criterion No. 3

Results will be evaluated in two stages, one for -q and the other for +q. As in the previous chapter, values of K' are chosen and then K is evaluated. For the maximum sector, K should equal K'.

(a) $G(s) = \frac{1 - e^{-sT}}{s(s + 1)(s + 2)}$ T = 1 sec.

1.) $q < 0$ (neg. sign accounted for in (6.4))

Substituting (2.11) and (4.5) into (6.4) and setting it equal to zero yields

$$(x + 4.68)(x + 1.73) + .5K(x - 4.54)(x + .295) + \frac{2qK(-1.508x^2 - 2.178x - .67)}{x + 1} + \frac{2qK(x^2 + 6.41x + 8.12)}{K'(x + 1)} = 0 \quad (6.5)$$

i) Try K' = 6.77

(6.5) becomes

$$(x + 4.68)(x + 1.73) + .5K(x - 4.54)(x + .295)$$

$$- \frac{2.72qK(x^2 + .897x - .39)}{x + 1} = 0 \qquad (6.6)$$

For q = 0, the root locus diagram of (6.6) is found in Fig. 2.2. From Fig. 6.2 it can be seen that K cannot be greater than 6.77 (Tsypkin value).

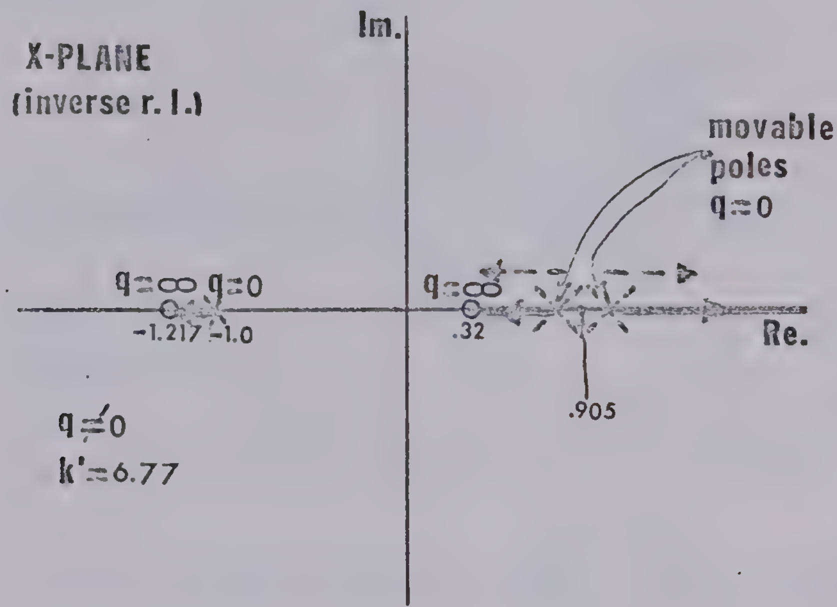

FIGURE 6.2 J. AND L. NO. 3 ROOT LOCUS $G(s)=\frac{1-e^{-s}}{s(s+1)(s+2)}$, A

2.) $q \geq 0$

Substituting (2.11) and (4.5) into (6.3) and setting the result equal to zero produces

$$(x + 4.68)(x + 1.73) + .5K(x - 4.54)(x + .295)$$

$$+ \frac{2qK(2.508x^2 - 2.068x - .67)}{x + 1}$$

$$+ \frac{2qK(x^2 + 6.41x + 8.12)}{K'(x + 1)} = 0. \qquad (6.7)$$

i) Try K' = 6.77

(6.7) becomes

$$(x + 4.68)(x + 1.73) + .5K(x - 4.54)(x + .295)$$

$$+ 5.312qK(x - .211 - j.395)(x - .211 + j.395) = 0. \qquad (6.8)$$

Once again the plot of the roots of (6.8) with $q = 0$ is found in Fig. 2.2. From Fig. 6.3 it can be seen that K can assume any value and a q can still be found that will take the roots of (6.8) off the positive real axis. Therefore K' is too low.

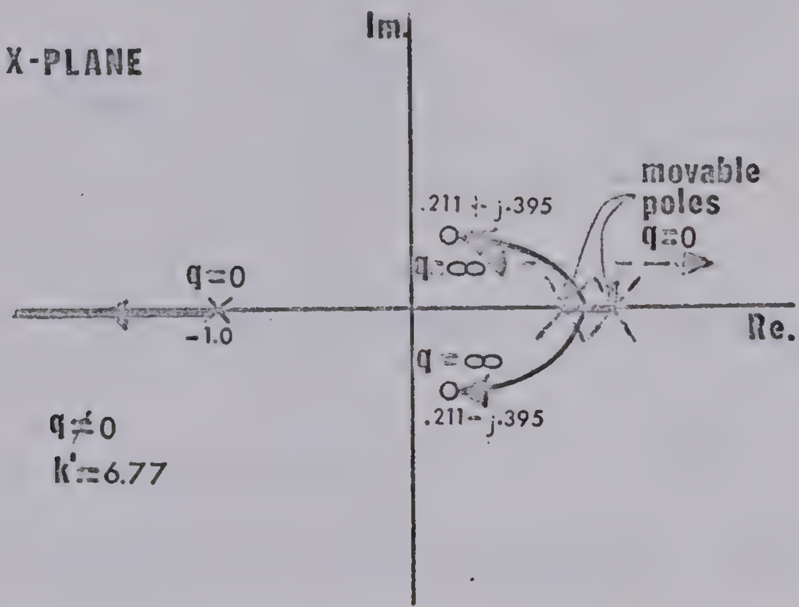

FIGURE 6.3 J. AND L. NO. 3 ROOT LOCUS $G(s)=\frac{1-e^{-s}}{s(s+1)(s+2)}$, B

ii) Try $K' = 10$

The same procedure as in part i yields

$$(x + 4.68)(x + 1.73) + .5K(x - 4.54)(x + .295)$$

$$+ \frac{5.216qK(x - .417)(x - .130)}{x + 1} = 0. \quad (6.9)$$

From Fig. 6.4 it can be seen that the maximum value of K possible and still find a q which will satisfy the criterion is the value of K at $x = .417$. At $x = .417$, $K = 7.46$. K' was chosen too high, since $K < K'$. It would now seem logical to evaluate K at the point where a value of K' places multiple zeros of (6.7) for $q \neq 0$ on the positive real axis. This value is found to be $K' = 9.53$.

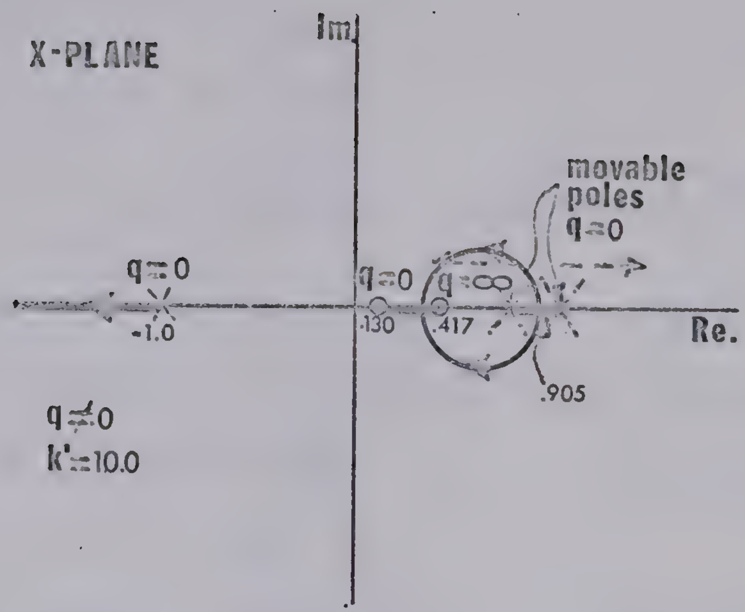

FIGURE 6.4 J. AND L. NO. 3 ROOT LOCUS $G(s)=\frac{1 - e^{-s}}{s(s+1)(s+2)}$, C

$K' = 9.53$ in (6.7) produces

$$(x + 4.68)(x + 1.73) + .5K(x - 4.54)(x + .295)$$

$$+ \frac{5.22qK(x - .261)^2}{x + 1} = 0. \qquad (6.10)$$

K at $x = .261$ is 8.27. To satisfy the criterion, q must equal infinity. Therefore, for $K' \geq 9.53$, $K < 8.27$ and for $K' < 9.53$, $K < \infty$. Therefore, for $K' = 9.53$, $K < 9.53$.

$$\text{(b)} \quad G(s) = \frac{1 - e^{-sT}}{s^2(s + 1)} \quad T = 1 \text{ sec.}$$

1.) $q < 0$

Substituting (2.15) and (4.8) into (6.4) and setting the result equal to zero yields

$$(x + 4.67) - 1.5K(x + .119) + \frac{2qK(-.5x^2 - .6775x - .178)}{x + 1}$$

$$+ \frac{2qK(x + 4.67)}{K'(x + 1)} = 0. \qquad (6.11)$$

i) Try $K' = .667$ (Tsypkin value)

(6.11) becomes

$$(x + 4.67) - 1.5K(x + .119) - \frac{2qK(.5x^2 - .8225x - 6.88)}{x + 1} = 0. \quad (6.12)$$

(6.12) approaches $-\infty$ as x approaches ∞, for $K > .667$. Therefore for negative q the criterion produces the same result as the Tsypkin case.

2.) $q \geq 0$

Substituting (2.15) and (4.8) into (6.3) which is set equal to zero yields

$$(x + 4.67) - 1.5K(x + .119) + \frac{2qK(.5x^2 - 2.32x - .178)}{x + 1} + \frac{2qK(x + 4.67)}{K'(x + 1)} = 0. \quad (6.13)$$

i) Try $K' = .667$

$$(x + 4.67) - 1.5K(x + .119) + \frac{qK(x^2 - 1.645x + 13.64)}{x + 1} = 0. \quad (6.14)$$

Fig. 2.4 is the root locus plot of the poles of (6.14) with q as the variable. From Fig. 6.5, the root locus plot of (6.14), it can be seen that $K' = .667$ is too low as K can be made any value and a q can be found such that the criterion is satisfied. As in the previous example the optimum value of K' is the value which places both zeros coincident on the positive real axis. This value is found to be $K' = 2.36$. Substituting $K' = 2.36$ into (6.13) we obtain

$$(x + 4.67) - 1.5K(x + .119) + \frac{qK(x - 1.9)^2}{x + 1} = 0. \quad (6.15)$$

Fig. 6.6 is the root locus plot of (6.15). $K = 2.16$ at

$x = 1.9$. Therefore for $K' \geq 2.36$, $K < 2.16$ and for $K' < 2.36$, $K < \infty$. Therefore for $K' = 2.36$, $K < 2.36$ and $q = \infty$.

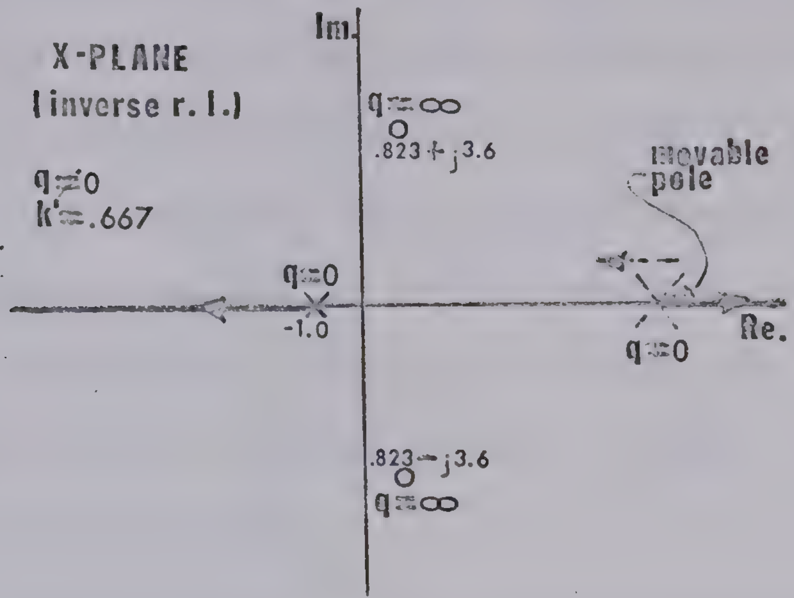

FIGURE 6.5 J. AND L. NO. 3 ROOT LOCUS $G(s) = \dfrac{1 - e^{-s}}{s^2(s + 1)}$, A

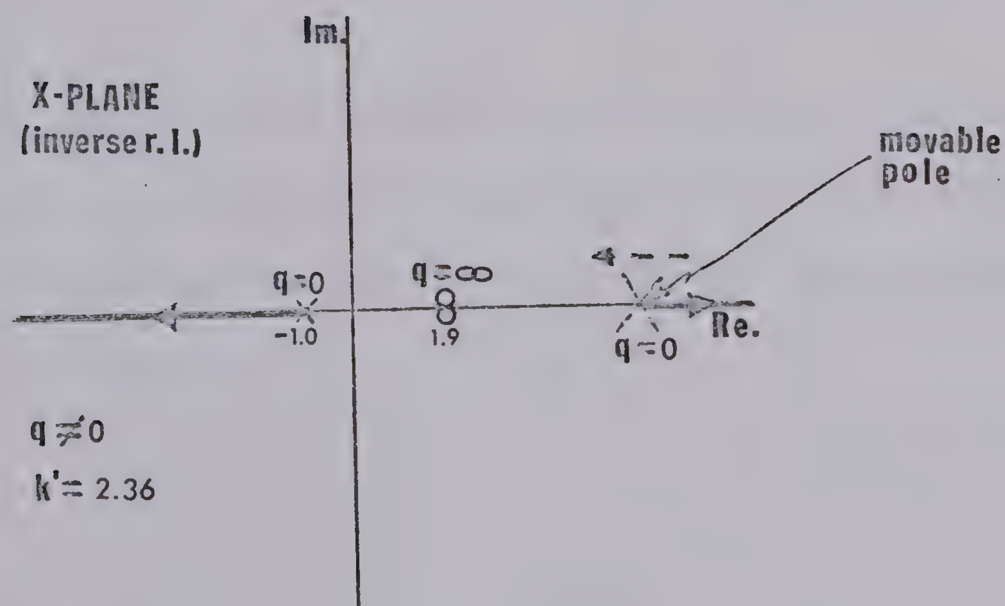

FIGURE 6.6 J. AND L. NO. 3 ROOT LOCUS $G(s) = \dfrac{1 - e^{-s}}{s^2(s + 1)}$, B

CHAPTER 7 CONCLUSION

A root locus method has been developed which makes the application of the stability criteria of Tsypkin and Jury and Lee practical without the aid of a digital computer. The Tsypkin and the first Jury and Lee criterion are the easiest to apply, resulting in the desired answer in a straight forward manner, while the second and third Jury and Lee criteria require closing in on the right value of K' to obtain the maximum sector.

An advantage of the root locus method is that it enables one to observe how rapidly the sector width is changing for various values of K', and it is possible to gain an insight into the general frequency plot of the system under investigation.

All results were obtained using slide rule accuracy. In the appendix are Fortran IV programs, which were developed to confirm results and to furnish a means of obtaining greater accuracy if desired.

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APPENDIX COMPUTER PROGRAMS OF THE CRITERIA

The following three programs evaluate the Tsypkin and Jury and Lee criteria by digital computer to an accuracy of three decimal places. Each of the three programs gives the Tsypkin value and the equivalent Jury and Lee result. The programs will evaluate systems which have the following general form of linear plant and zero order hold.

$$G(z) = \frac{Gz^N(z - A)(z - B)(z - C)(z - D)}{(z - O)(z - P)(z - R)(z - S)(z - U + jV)(z - U + jV)}$$

The programs enable any sampling period T to be used. For example, for the plant $G(z) = \frac{.368(z + .717)}{(z - 1)(z - .368)}$ and T = 1 sec. the values to be used in a program are T = 1.0, G = .368, N = 1, A = -.717, B = 0.0, C = 0.0, D = 0.0, O = 1.0, P = .368, R = 0.0, S = 0.0, U = 0.0, and V = 0.0.

A Jury and Lee No. 1

The value of the maximum negative slope gain encountered is specified as ZKD in the program. In this case ZKD = 0.0.

```
C       RESULTS FOR JURY & LEE #1
        DIMENSION AAA(3000),BBB(3000),CCC(3000)
        T=1.0
        NN=(3.141592*2.0)/(T*.015707963)
        W=.001
        DO 1 I=1,NN
        X=COS(W*T)
        Y=SIN(W*T)
        ZKD=0.0
        G=.3
        N=1
        A=-1.0/3.0
        B=0.0
        C=0.0
        D=0.0
        O=0.0
        P=1.0
```

```
      R=0.0
      S=0.0
      U=0.0
      V=0.0
      Z=N
      BM=SQRT(((X-A)**2)+(Y**2))
      CM=SQRT(((X-B)**2)+(Y**2))
      DM=SQRT(((X-C)**2)+(Y**2))
      EM=SQRT(((X-D)**2)+(Y**2))
      FM=SQRT(((X-O)**2)+(Y**2))
      GM=SQRT(((X-P)**2)+(Y**2))
      HM=SQRT(((X-R)**2)+(Y**2))
      OM=SQRT(((X-S)**2)+(Y**2))
      PM=SQRT(((X-U)**2)+((Y+V)**2))
      QM=SQRT(((X-U)**2)+((Y-V)**2))
      AAB=0.0
      IF(G.LT.0.0)AAB=3.141592
      AA=W*Z*T
      BA=ATAN2(Y,X-A)
      CA=ATAN2(Y,X-B)
      DA=ATAN2(Y,X-C)
      EA=ATAN2(Y,X-D)
      FA=ATAN2(Y,X-O)
      GA=ATAN2(Y,X-P)
      HA=ATAN2(Y,X-R)
      OA=ATAN2(Y,X-S)
      PA=ATAN2(Y+V,X-U)
      QQA=ATAN2(Y-V,X-U)
      GMM=(ABS(G)*BM*CM*DM*EM)/(FM*GM*HM*OM*PM*QM)
      GMA=AAB+AA+BA+CA+DA+EA-FA-GA-HA-OA-PA-QQA
      AAA(I)=GMM*COS(GMA)
      BB=GMM*SIN(GMA)
      BBC=(X-1.0)*AAA(I)-Y*BB
      BBB(I)=AAA(I)*(1.0-X)-BB*Y
      CCD=Y*AAA(I)+BB*(X-1.0)
      CCC(I)=(BBC**2+CCD**2)/2.0
    1 W=W+.015707963/2.0
      FG=1000.0
      DO 2 J=1,NN
      IF(FG.LE.AAA(J))GO TO 2
      FG=AAA(J)
    2 CONTINUE
      ZKT=-1.0/FG
      IF(ZKT.GE.0.0)GO TO 30
      WRITE(6,706)
      GO TO 11
   30 CONTINUE
      WRITE(6,700)ZKT
    9 CONTINUE
      Q=0.0
      QA=1000.0
      QB=100.0
      QC=10.0
      FL=-1000.0
```

```
    6 CONTINUE
      FG=1000.0
      DO 3 K=1,NN
      SDJL=AAA(K)+Q*(BBB(K)-ZKD*CCC(K))
      IF(FG.LE.SDJL) GO TO 3
      FG=SDJL
      SD=FG
    3 CONTINUE
      IF(SD.LE.FL) GO TO 4
      FL=SD
      SDN=FL
      QQ=Q
    4 CONTINUE
      Q=Q+QB
      IF(Q.GT.QA) GO TO 5
      GO TO 6
    5 FGG=-1.0/SDN
      IF(QB.EQ.10.0)QC=1.0
      IF(QB.EQ.1.0)QC=.1
      IF(QB.EQ..1)QC=.01
      IF(QB.EQ..01)QC=.001
      IF(QB.EQ..001)GO TO 7
      IF(QQ.EQ.0.0)QQ=QQ+QB
      Q=QQ-QB
      IF(Q.EQ.0.0)QQ=QQ-QB
      QA=QQ+QB
      QB=QC
      FL=-1000.0
      GO TO 6
    7 ZK=-1.0/SDN
      WRITE(6,704)ZK,QQ
      WRITE(6,705)ZKD
   11 STOP
  700 FORMAT(10X,20HTHE TSYPKIN VALUE ISF9.3)
  704 FORMAT(1X,18HTHE SECTOR GAIN ISF9.3,1X,10HFOR A Q OFF9.3)
  705 FORMAT(1X,36HAND A MAXIMUM NEGATIVE SLOPE GAIN OFF9.3)
  706 FORMAT(1X,31HTHE TSYPKIN VALUE IS K=INFINITY)
      END
```

B Jury and Lee No. 2

```
C     RESULTS FOR JURY & LEE #2
      DIMENSION AAA(3000),BBB(3000),CCC(3000)
      T=1.0
      NN=(3.141592*2.0)/(T*.015707963)
      W=.001
      DO 1 I=1,NN
      X=COS(W*T)
      Y=SIN(W*T)
      G=.368
      N=1
```

```
      A=-.717
      B=0.0
      C=0.0
      D=0.0
      O=1.0
      P=.368
      R=0.0
      S=0.0
      U=0.0
      V=0.0
      Z=N
      BM=SQRT(((X-A)**2)+(Y**2))
      CM=SQRT(((X-B)**2)+(Y**2))
      DM=SQRT(((X-C)**2)+(Y**2))
      EM=SQRT(((X-D)**2)+(Y**2))
      FM=SQRT(((X-O)**2)+(Y**2))
      GM=SQRT(((X-P)**2)+(Y**2))
      HM=SQRT(((X-R)**2)+(Y**2))
      OM=SQRT(((X-S)**2)+(Y**2))
      PM=SQRT(((X-U)**2)+((Y+V)**2))
      QM=SQRT(((X-U)**2)+((Y-V)**2))
      AAB=0.0
      IF(G.LT.0.0)AAB=3.141592
      AA=W*Z*T
      BA=ATAN2(Y,X-A)
      CA=ATAN2(Y,X-B)
      DA=ATAN2(Y,X-C)
      EA=ATAN2(Y,X-D)
      FA=ATAN2(Y,X-O)
      GA=ATAN2(Y,X-P)
      HA=ATAN2(Y,X-R)
      OA=ATAN2(Y,X-S)
      PA=ATAN2(Y+V,X-U)
      QQA=ATAN2(Y-V,X-U)
      GMM=(ABS(G)*BM*CM*DM*EM)/(FM*GM*HM*OM*PM*QM)
      GMA=AAB+AA+BA+CA+DA+EA-FA-GA-HA-OA-PA-QQA
      AAA(I)=GMM*COS(GMA)
      BB=GMM*SIN(GMA)
      BBB(I)=(X-1.0)*AAA(I)-Y*BB
      CCD=Y*AAA(I)+BB*(X-1.0)
      CCC(I)=(BBB(I)**2+CCD**2)/2.0
    1 W=W+.015707963/2.0
      FG=1000.0
      DO 2 J=1,NN
      IF(FG.LE.AAA(J))GO TO 2
      FG=AAA(J)
    2 CONTINUE
      ZKT=-1.0/FG
      IF(ZKT.GE.0.0)GO TO 30
      WRITE(6,706)
      GO TO 11
   30 CONTINUE
```

```
      WRITE(6,700)ZKT
      ZC=1.0
      ZKD=ZKT
    9 CONTINUE
      ZA=ZC
      Q=0.0
      QA=1000.0
      QB=100.0
      QC=10.0
      FL=-1000.0
    6 CONTINUE
      FG=1000.0
      DO 3 K=1,NN
      SDJL=AAA(K)+Q*(BBB(K)-ZKD*CCC(K))
      IF(FG.LE.SDJL) GO TO 3
      FG=SDJL
      SD=FG
    3 CONTINUE
      IF(SD.LE.FL) GO TO 4
      FL=SD
      SDN=FL
      QQ=Q
    4 CONTINUE
      Q=Q+QB
      IF(Q.GT.QA) GO TO 5
      GO TO 6
    5 FGG=-1.0/SDN
      IF(QB.EQ.10.0)QC=1.0
      IF(QB.EQ.1.0)QC=.1
      IF(QB.EQ..1)QC=.01
      IF(QB.EQ..01)QC=.001
      IF(QB.EQ..001)GO TO 7
      IF(QQ.EQ.0.0)QQ=QQ+QB
      Q=QQ-QB
      IF(Q.EQ.0.0)QQ=QQ-QB
      QA=QQ+QB
      QB=QC
      FL=-1000.0
      GO TO 6
    7 ZK=-1.0/SDN
      IF(ZK.LE.ZKD)GO TO 8
   10 ZKD=ZKD+ZA
      GO TO 9
    8 IF(ZK.LE.0.0)GO TO 10
      IF(ZA.EQ..0001)ZK=ZKD
      IF(ZA.EQ..0001)WRITE(6,704)ZK,QQ
      IF(ZA.EQ..0001)WRITE(6,705)ZKD
      IF(ZA.EQ..0001)GO TO 11
      ZKD=ZKD-ZA
      IF(ZA.EQ.1.0)ZC=.1
      IF(ZA.EQ..1)ZC=.01
      IF(ZA.EQ..01)ZC=.001
      IF(ZA.EQ..001)ZC=.0001
      GO TO 9
```

```
 11 STOP
700 FORMAT(10X,20HTHE TSYPKIN VALUE ISF9.3)
704 FORMAT(1X,18HTHE SECTOR GAIN ISF9.3,1X,10HFOR A Q OFF9.3)
705 FORMAT(1X,27HAND A MAXIMUM SLOPE GAIN OFF9.3)
706 FORMAT(1X,21HTHE TSYPKIN VALUE IS K=INFINITY)
    END
```

C Jury and Lee No. 3

```
C       RESULTS FOR JURY & LEE #3
        DIMENSION AAA(3000),BBB(3000),CCC(3000),DDD(3000)
        T=1.0
        NN=(3.141592*2.0)/(T*.01507963)
        W=.001
        DO 1 I=1,NN
        X=COS(W*T)
        Y=SIN(W*T)
        G=.368
        N=1
        A=-.717
        B=0.0
        C=0.0
        D=0.0
        O=1.0
        R=.368
        P=0.0
        S=0.0
        U=0.0
        V=0.0
        Z=N
        BM=SQRT(((X-A)**2)+(Y**2))
        CM=SQRT(((X-B)**2)+(Y**2))
        DM=SQRT(((X-C)**2)+(Y**2))
        EM=SQRT(((X-D)**2)+(Y**2))
        FM=SQRT(((X-O)**2)+(Y**2))
        GM=SQRT(((X-P)**2)+(Y**2))
        HM=SQRT(((X-R)**2)+(Y**2))
        OM=SQRT(((X-S)**2)+(Y**2))
        PM=SQRT(((X-U)**2)+((Y+V)**2))
        QM=SQRT(((X-U)**2)+((Y-V)**2))
        AAB=0.0
        IF(G.LT.0.0)AAB=3.141592
        AA=W*Z*T
        BA=ATAN2(Y,X-A)
        CA=ATAN2(Y,X-B)
        DA=ATAN2(Y,X-C)
        EA=ATAN2(Y,X-D)
        FA=ATAN2(Y,X-O)
        GA=ATAN2(Y,X-P)
        HA=ATAN2(Y,X-R)
        OA=ATAN2(Y,X-S)
        PA=ATAN2(Y+V,X-U)
```

```
      QQA=ATAN2(Y-V,X-U)
      GMM=(ABS(G)*BM*CM*DM*EM)/(FM*GM*HM*OM*PM*QM)
      GMA=AAB+AA+BA+CA+DA+EA-FA-GA-HA-OA-PA-QQA
      AAA(I)=GMM*COS(GMA)
      BB=GMM*SIN(GMA)
      BBB(I)=AAA(I)*(1.0-X)-Y*BB
      CCC(I)=1.0-X
      DDD(I)=AAA(I)*(X-1.0)-Y*BB
    1 W=W+.015707963/2.0
      FG=1000.0
      DO 2 J=1,NN
      IF(FG.LE.AAA(J))GO TO 2
      FG=AAA(J)
    2 CONTINUE
      ZKT=-1.0/FG
      IF(ZKT.GE.0.0)GO TO 30
      WRITE(6,706)
      GO TO 26
   30 CONTINUE
      WRITE(6,700)ZKT
      ZZZ=-1.0
   10 ZKD=ZKT
      ZC=1.0
    9 CONTINUE
      ZA=ZC
      Q=0.0
      QA=1000.0
      QB=100.0
      QC=10.0
      FL=-1000.0
    6 CONTINUE
      FG=1000.0
      DO 3 K=1,NN
      IF(ZZZ.GE.0.0)GO TO 20
      SDJL=AAA(K)+Q*((BBB(K))+(CCC(K)/ZKD))
      GO TO 21
   20 SDJL=AAA(K)-Q*((DDD(K))-(CCC(K)/ZKD))
   21 CONTINUE
      IF(FG.LE.SDJL) GO TO 3
      FG=SDJL
      SD=FG
    3 CONTINUE
      IF(SD.LE.FL) GO TO 4
      FL=SD
      SDN=FL
      QQ=Q
    4 CONTINUE
      Q=Q+QB
      IF(Q.GT.QA) GO TO 5
      GO TO 6
    5 FGG=-1.0/SDN
      IF(QB.EQ.10.0)QC=1.0
```

```
      IF(QB.EQ.1.0)QC=.1
      IF(QB.EQ..1)QC=.01
      IF(QB.EQ..01)QC=.001
      IF(QB.EQ..001)GO TO 7
      IF(QQ.EQ.0.0)QQ=QQ+QB
      Q=QQ-QB
      IF(Q.EQ.0.0)QQ=QQ-QB
      QA=QQ+QB
      QB=QC
      FL=-1000.0
      GO TO 6
    7 ZK=-1.0/SDN
      IF(ZK.LE.ZKD)GO TO 8
   50 ZKD=ZKD+ZA
      GO TO 9
    8 IF(ZK.LT.0.0)GO TO 50
      IF(ZA.EQ..0001)GO TO 11
      ZKD=ZKD-ZA
      IF(ZA.EQ.1.0)ZC=.1
      IF(ZA.EQ..1)ZC=.01
      IF(ZA.EQ..01)ZC=.001
      IF(ZA.EQ..001)ZC=.0001
      GO TO 9
   11 IF(ZZZ.GE.0.0)GO TO 23
      QQ1=QQ
      ZKD1=ZKD
      ZK1=ZKD1
      ZZZ=1.0
      GO TO 10
   23 QQ2=QQ
      ZKD2=ZKD
      ZK2=ZKD2
      EEE=ZK1-ZK2
      IF(EEE)24,25,25
   25 WRITE(6,704)ZK1,QQ1
      WRITE(6,705)ZKD1
      GO TO 26
   24 QQ2=-QQ2
      WRITE(6,704)ZK2,QQ2
      WRITE(6,705)ZKD2
   26 CONTINUE
      STOP
  700 FORMAT(10X,20HTHE TSYPKIN VALUE ISF9.3)
  704 FORMAT(1X,18HTHE SECTOR GAIN ISF9.3,1X,10HFOR A Q OFF9.3)
  705 FORMAT(1X,27HAND A MAXIMUM SLOPE GAIN OFF9.3)
  706 FORMAT(1X,31HTHE TSYPKIN VALUE IS K=INFINITY)
      END
```